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Algebra 1

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Quiz #2

Problems :

1. Let G be the matrix group $\mathrm{GL}(n,\mathbb{C})$ of all $n\times n$ matrices A with complex entries and $\det(A)\neq 0$. This *is* a group under matrix multiplication, and so is the subgroup $N=\mathrm{SL}(n,\mathbb{C})$ of matrices with determinant $+1$.
   (a) Prove that $\mathrm{SL}(n,\mathbb{C})$ is a normal subgroup of $\mathrm{GL}(n,\mathbb{C})$. What does this implies for the quotient $\mathrm{GL}(n,\mathbb{C})/\mathrm{SL}(n,\mathbb{C})$?
   (b) Let
   $$\begin{array}{rlcl}\tilde{det}: & \mathrm{GL}(n,\mathbb{C})/\mathrm{SL}(n,\mathbb{C}) & \to & \mathbb{C}^\times \\ & M\mathrm{SL}(n,\mathbb{C}) & \mapsto & det(M)\end{array}$$
   Prove that $\tilde{det}$ is a well defined isomorphism of groups. (Be careful you have to prove several properties here.)

   Solution : *We claim that N is normal in G, and that the quotient group G/N is isomorphic to the group $(\mathbb{C}^\times,\cdot)$ of nonzero complex numbers under multiplication.*
   Normality of N follows because the determinant has the properties
   $$\det I=1 \qquad \det(AB)=\det(A)\cdot\det(B) \qquad \det(A^{-1})=\frac{1}{\det(A)}$$
   If $A\in G$ and $B\in N$ we get $\det(ABA^{-1})=\det(A)\det(B)\det(A)^{-1}=\det(B)=1$, which shows that $ANA^{-1}\subseteq N$ for all $A\in G$. Thus N is normal. The determinant map $\phi(A)=\det A$ is a natural homomorphism $\phi:\mathrm{GL}(n,\mathbb{C})\to\mathbb{C}^\times$. It is a group homomorphism because the determinant is multiplicative, and it is surjective because if $\lambda\neq 0$ in $\mathbb{C}$ the diagonal matrix $D=\mathrm{diag}(\lambda^{1/n},\ldots,\lambda^{1/n})$ has $\det D=\lambda$. (Here $\lambda^{1/n}$ is any complex n^{th} root of λ ; for instance if λ has polar form $\lambda=re^{i\theta}$ we can take the principal n^{th} root $\lambda^{1/n}=r^{1/n}e^{i\theta/n}$ where $r^{1/n}$ is the usual n^{th} root of a non-negative real number.) The kernel of ϕ is precisely $N=\mathrm{SL}(n,\mathbb{C})$, by definition of $\mathrm{SL}(n,\mathbb{C})$. The conditions of the First Isomorphism Theorem are fulfilled. We conclude that $\mathrm{GL}(n,\mathbb{C})/\mathrm{SL}(n,\mathbb{C})\cong(\mathbb{C}^\times,\cdot)$ as claimed.

2. (a) Give the center of S_3. What you can deduce about S_3 ?
   $$S_3=\{Id,(1,2),(1,3),(2,3),(1,2,3),(1,3,2)\}$$
   $(1,2)(1,2,3)=(1,3)$, and $(1,2,3)(1,2)=(2,3)$, so neither $(1,2)$ nor $(1,2,3)$ is in the center.
   $(2,3)(1,3,2)=(1,3)$, and $(1,3,2)(2,3)=(1,2)$, so neither $(2,3)$ nor $(1,3,2)$ is in the center.
   $(1,2)(1,3)=(1,2,3)$, and $(1,3)(1,2)=(1,3,2)$, so $(1,3)$ isn't in the center either.
   That leaves the identity permuation (1), which has to commute with everything, so the center is just $\{(1)\}$.
   (b) Give the order of $(1,3,2)$ in S_3 and the group generated by $(1,3,2)$, to which well know group is it isomorphic to ?
   $(1,2,3)^2=(1,3,2)$, and $(1,2,3)^3=Id$, so $<(1,2,3)>=\{Id,(1,2,3),(1,3,2)\}$.
   $o(1,2,3)=3$ and $<(1,2,3)>\simeq\mathbb{Z}/3\mathbb{Z}$.

3. For each the following pair of groups, decide whether they are isomorphic or not. In each case, give a brief reason why.
   (a) U_5 and U_{10}.
   Yes. They are both cyclic of order 4.
   (b) U_8 and $\mathbb{Z}/4\mathbb{Z}$.
   No. U_8 doesn ?t has an element of order 4, but $\mathbb{Z}/4\mathbb{Z}$ does.
   (c) U_{10} and $\mathbb{Z}/4\mathbb{Z}$.
   Yes. They are both cyclic of order 4.
   (d) S_3 and $\mathbb{Z}/6\mathbb{Z}$.
   No. S_3 is not abelian, but $\mathbb{Z}/6\mathbb{Z}$ is.
4. Give the order of $[2]$ in the group $\mathbb{Z}/6\mathbb{Z}$, give the subgroup of $\mathbb{Z}/6\mathbb{Z}$ generated by $[2]$. To which well-know group is it isomorphic to ?
   Be careful we are in additive notation here ! ! $2.[2] = [4]$, $3.[2] = [0]$. then $o([2]) = 3$ and $< [2] >= \{[0], [2], [4]\} \simeq \mathbb{Z}/3\mathbb{Z}$.
5. Give the order of $[5]$ in U_6, give the subgroup of U_6, generated by $[5]$. To which well-know group is it isomorphic to ?
   Be careful we are in multiplicative notation here ! !
   $[5]^2 = [25] = [1]$. then $o([5]) = 2$ and $< [5] >= \{[1], [5]\} \simeq \mathbb{Z}/2\mathbb{Z}$.